Some Mysteries of Arithmetic Explained:
Secrets revealed that may help parents and teachers clarify mathematics for youngsters

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Winner of the Outstanding Professor Award from San Jose State University,
founded before the Civil War in 1857 as the oldest public university in California.
SJSU has 30,000 students and a faculty of 1,800 Ph.Ds.

Have you noticed that information presented in school and in textbooks is perceived by students as much too "sacred" to be touched? For example, how many students would have done this? Using a book borrowed from the library, a young teenager studied trigonometry on his own because he wanted to be a scientist some day. But the names of trig functions printed in the textbook did not make sense to him. So he created his own names! It worked beautifully until he earned a doctorate in physics and began working with other physicists who did not know what he was talking about. Whoa! Time to go back and memorize the textbook labels—which he did in a flash. That young man was Dr. Richard P. Feynman who won a Nobel prize in physics.

Even testing to assess "learning" implies that students must duplicate information in their brain without editing. The more exact the replication by the student's memory, the higher the grade. Should we encourage students to doodle with information instead of merely duplicating information in memory to pass a test?

A closer look at doodling

Only when one advances to a master's or doctorate degree, do the rules change from duplicating information to the discovery of information. All the way up to graduate school, the model is that of monks in a monastery carefully copying sacred scripture exactly as is. Then suddenly, one is in graduate school where the rules change. Students are now like people in a "think tank" attempting to decipher mysteries of the atom or speculating about the future in technology. By that time it is too late for most students. Their stunning success in school has been a by-product of "by heart" learning which is signaled by cliches such as "cramming for a test."

My premise in this article is that the discovery mode should be declassified from "only for the eyes of advanced students" to "here is an exciting toy that elementary students are fully capable of playing with and perhaps coming up with a breakthrough—especially since children now have access in their homes to the most powerful toy of the 21st century, the computer." Young people also have something that is a luxury for their parents—time to explore mysteries.

Some fun examples from arithmetic

Here is a fun example from arithmetic — usually too sacred to tinker with. Mathematics is, after all, a field of "absolute truths that have been proven for hundreds of years" or has it?

Only when one becomes an advanced student is the following carefully guarded secret revealed: Everything in mathematics is controversial including "self-evident" and "obvious" truths such as 2 + 2 = 4 and (- 2) + (- 2) = - 4. In mathematics, nothing is absolute including the concept of a “proof.” Sir Bertrand Russell and Alfred North Whitehead co-wrote **Principia Mathematica**, a prize-winning volume exploring the implications of 1 + 1 = 2. It turns out, for instance, that the numbers 1 and 2 have different behavior from other numbers such as 3, 4 and 5 (See Footnote 1). Russell is quoted as saying, “...mathematics may be defined as the subject where we do not know what we are talking about, neither do we know if what we are saying is true.”

Show me an exciting example

Like everyone else, I always believed that there was only one algebra and that was the algebra I learned in school. Laurie Buxton in a revealing book, **Mathematics for Everyone**, convinced me otherwise with the statement that there are many algebras, one of which by the Irish mathematician, Roland Hamilton, was used by Albert Einstein to predict the location of Mercury. Not only that, but Buxton suggested that anyone is capable of creating a new algebra by selecting some symbols and a few rules. The test is whether the new algebra is self-consistent, meaning there are no internal contradictions. More about this in a moment.

My attempt to create a new arithmetic that will explain some mysteries

I confess to you that I am not sure what the difference is between arithmetic and algebra. It looks as if one uses numbers and the other uses letters of the alphabet. But, that isn’t right either since arithmetic and algebra both use numbers and letters, but arithmetic does seem to use mostly numbers and algebra uses mostly letters.

So, I decided to play with something that looks simple such as arithmetic. Are there alternatives to arithmetic other than the model presented to us in school. Yes, I think so. But why fool around with an alternate model? What’s wrong with the standard school model? It seems to work. Or does it?

Doodling with the arithmetic we learned in school

I discovered that the standard arithmetic we learned in school has some severe limitations that are hidden until one begins to doodle with arithmetic. Here are some examples:

Multiplication is nothing more than repeated addition

With utmost confidence, teachers present to their students this premise: Multiplication is repeated addition. If this premise is true, I believe it only holds for whole positive numbers. It certainly does not explain negative numbers or fractions. Let me show you how I arrive at this conclusion.

Let’s start with positive numbers

2 plus 2 = + 4 and 2 times 2 = + 4 So far, so good.

Now, if multiplication is nothing more than repeated addition, then it follows that,

(- 2) plus (- 2) = - 4 and (- 2) times (- 2) = - 4

But, wait!

(- 2) times (- 2) does not equal - 4, but rather + 4.

Students are logical and ask for an explanation. I have yet to hear a satisfactory answer. A “satisfactory answer” is one that is received with a tiny voice in our head that says, “Yep! That makes perfect sense!”

W. H. Auden expressed his impatience with the mystery of (-2) times (-2) = + 4, this way:

“Minus times minus is a plus; the reason for this we need not discuss.”

An alternative model that explains the mystery

The school's interpretation of multiplication is that it is simply "repeated addition," but I have demonstrated that this may not be true for negative numbers. Here is a new interpretation of multiplication which will explain positive numbers, negative numbers and, as a bonus, it will also explain fractions.

New interpretation of addition and multiplication

Addition means to ***copy*** the first number, ***copy*** the second number and combine like this:

2 + 2 = 4

Multiplication is <u>not</u> the same as addition because the first number is <u>not</u> a "real" number, but an instruction to ***copy*** a certain number of times the second number which is "real" and then add. For example,

Application to positive numbers

2 times 2 means to ***copy*** 2 twice and then add like this:

2 plus 2 = + 4.

3 times 2 means to ***copy*** 2 three times and then add like this:

2 plus 2 plus 2 = + 6

4 times 2 means to ***copy*** 2 four times and then add like this:

2 plus 2 plus 2 plus 2 = + 8

Note: The first number in a pair to be multiplied is the instruction to ***copy*** the second number that is a "real" number. (Keep in mind that arithmetic can only be performed with a *pair of numbers*. For example, you cannot add 3 + 4 + 5. It is impossible! Here is what you <u>can</u> do: You can add 3 + 4 = 7 and then add 7 + 5 = 12. Another option: Add 3 + 5 = 8 and then add 8 + 4 = 12.)

Application to negative numbers

- 2 times + 2 means to ***copy*** + 2 twice and add like this:

2 + 2 = + 4. The negative sign on the first number tells us to reverse the direction of the product. So, + 4 becomes - 4.

- 3 times 2 means to ***copy*** + 2 thrice and add like this:

2 + 2 + 2 = + 6. The negative sign on the first number tells us to reverse the direction of the product. So, 6 becomes - 6 - 4 times + 2 means to ***copy*** + 2 four times and add like this:

2 + 2 + 2 + 2 = + 8. The negative sign on the first number tells us to reverse the direction of the product. So, 8 becomes - 8.

But, what happens when we reverse the numbers in the examples above? (This test is called the **commutative property** of multiplication, often written as ab = ba.)

2 times - 2 tells us to ***copy*** - 2 twice and add like this:

(- 2) + (- 2) = - 4

2 times - 3 Tells us to ***copy*** - 3 twice and add like this:

(- 3) + (- 3) = - 6

2 times - 4 =, Tells us to ***copy*** - 4 twice and add like this:

(- 4) + (- 4) = - 8

CONCLUSION

Whether you multiply, for instance, - 2 times + 2 or reverse the order and multiply + 2 times - 2 the product is the same. In this case, - 4.

Let's kick it up a notch and add a little complexity

- 2 times - 2 means to copy -2 twice and add like this:

(- 2) + (- 2) = - 4. The negative sign on the first number tells us to reverse direction of the product. So, - 4 becomes + 4.

- 3 times - 2 means to ***copy*** - 2 three times and add like this:

(- 2) + (- 2) + (- 2) = - 6. The negative sign on the first number tells us to reverse direction of the product. So, - 6 becomes + 6.

Division of whole positive numbers

Here is how my new interpretation of multiplication applies to division:

$\frac{4}{2}$ asks how many times must we ***copy*** 2 and add to eliminate 4?

2 + 2 = 4 So the answer is 2.

$\frac{6}{2}$ asks how many times must we ***copy*** 2 and add to eliminate 6:

2 + 2 + 2 = 6 So the answer is 3.

$\frac{6}{3}$ asks how many times must we ***copy*** 3 and add to eliminate 6?

3 + 3 = 6 So the answer is 2.

Adding fractions

Let's test whether the traditional school interpretation that "multiplication is repeated addition" holds true for fractions.

Let's start with:

Example 1: $\frac{1}{2}$ plus $\frac{1}{4}$ equals (We need a common denominator to add.)

$\frac{2}{4}$ plus $\frac{1}{4}$ equals $\frac{3}{4}$ = 75 percent (3 divided by 4 = .75 *or* 75 percent)

Why do we need a common denominator to add fractions but it is not needed to multiply fractions is yet another mystery rarely explained to the satisfaction of students (See Footnote 2).

Example 2: $\frac{1}{2}$ plus $\frac{1}{3}$ equals (We need a common denominator to add.)

$\frac{3}{6}$ plus $\frac{2}{6}$ equals $\frac{5}{6}$ = 83 percent

Example 3: $\frac{1}{2}$ plus $\frac{1}{2}$ equals $\frac{2}{2}$ = 100 percent

CONCLUSION

If we start with a fraction such as $\frac{1}{2}$ and add an increment such as $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$, the result is a value larger than $\frac{1}{2}$. Intuitively, that makes sense. Adding something to something results in something bigger or larger than the $\frac{1}{2}$ we started with. That's the nature of addition.

So, since "multiplication is repeated addition," if we multiply the fractions above, the result should be an increase in value—something bigger or larger. Let's test to see whether this is true.

Multiplying fractions

Example 1: $\frac{1}{2}$ times $\frac{1}{2}$ equals $\frac{1}{4}$ = 25 percent (less than the 50 percent we started with)

As I mentioned before, we have another mystery: Why is it necessary to have a common denominator to add fractions, but not when multiplying fractions? After all, the claim is that multiplication is merely repeated addition.

Example 2: $\frac{1}{2}$ times $\frac{1}{4}$ equals $\frac{1}{8}$ = 12.5 percent (less than the 50 percent we started with)

Example 3: $\frac{1}{2}$ times $\frac{1}{8}$ equals $\frac{1}{16}$ = 6.25 percent (less than the 50 percent we started with)

CONCLUSION

If we multiply the initial fraction of $\frac{1}{2}$, with some increment, contrary to expectation, the result is a <u>decrease</u> in value.

Adding fractions <u>increases</u> the result but multiplying fractions <u>decreases</u> the result, exactly the opposite of what we would expect if "multiplication is actually repeated addition." Why did this happen? There is no satisfactory explanation forthcoming from standard school arithmetic.

Let's solve the mystery with an alternate interpretation of multiplication

$\frac{1}{2}$ times $\frac{1}{2}$ means to copy one-half of $\frac{1}{2}$ which is $\frac{1}{4}$ = 25 percent.

$\frac{1}{2}$ times $\frac{1}{4}$ means to copy one-half of $\frac{1}{4}$ which is $\frac{1}{8}$ = 12.5 percent.

$\frac{1}{2}$ times $\frac{1}{8}$ means to copy one-half of $\frac{1}{8}$ which is $\frac{1}{16}$ = 6.25 percent.

CONCLUSION

With my new copy rule, it makes "sense" that when one fraction is multiplied by a second fraction, there is a <u>decrease</u> in value.

Dividing fractions using standard school arithmetic: Another mystery to be solved

$$\frac{\frac{1}{2}}{\frac{1}{2}} =$$

Standard School Solution: $\frac{1}{2}$ multiplied by $\frac{2}{1}$ = 1

Children are logical and, if encouraged, will ask the following questions:

"Teacher, why did you turn the bottom fraction upside down and then multiply by the top fraction?"

"Teacher, instead of turning the bottom fraction upside down, will it work if I turn the top fraction upside down and multiply?"

"Teacher, will it work if I turn both the top and the bottom fractions upside down and multiply?"

"Teacher, we started with a problem in division and ended up multiplying. Why is that?"

There is a satisfactory explanation for each question which requires more detail than we have space for in this article. The curious reader can find the answer in my book: **Brainswitching: Learning on the right side of the brain**. See Chapter 9: Use brainswitching to learn the second most "difficult" subject in school.

Now, I would like to explain the division of fractions using the new copy rule for multiplication.

Application of the new rule to division with fractions

$\dfrac{\frac{1}{2}}{\frac{1}{2}}$ means to copy $\frac{1}{2}$ as many times as it takes to equal $\frac{1}{2}$. The answer is 1 since we copy $\frac{1}{2}$ once.

$\dfrac{\frac{2}{1}}{\frac{1}{2}}$ means to copy $\frac{1}{2}$ as many times as it takes to equal 2. The answer is 4 since $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=2.$

$\dfrac{\frac{4}{1}}{\frac{1}{2}}$ means to copy $\frac{1}{2}$ as many times as it takes to equal 4. The answer is 8 since

$$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=4$$

I selected simple examples so that the underlying pattern is transparent.

GRAND CONCLUSION

While algorithms that we all learned "by heart" in school enable us automatically and efficiently to apply arithmetic to solve for addition, subtraction, multiplication, and division, the explanation for why the procedures work is a mystery. For whole positive numbers I have demonstrated that "multiplication is simply repeated addition." However, the premise is ***false*** for negative numbers and fractions.

The solution to the mystery is a new copy rule for multiplication which is internally consistent—meaning there are no contradictions. I do not believe that the new copy rule is efficient for actual computation, but the new rule does solve some mysteries in arithmetic by explaining the "inner structure" of arithmetic with positive numbers, negative numbers and fractions.

"Inner structure" is a term I borrowed from the Gestalt psychologists who believed that understanding only comes from "insight" which is suddenly seeing a cause-effect relationship. Seeing a cause-effect pattern is the key to one-trial learning which they advocate rather than memorization by many repetitious trials as in "by heart" learning.

We now believe that one-trial (or first-trial) learning happens in the right brain while multiple-trials to learn "by heart" takes place in the left brain. The reason: The right brain is looking for a cause-effect pattern to explain something. The left brain is looking for flaws—reasons to filter or block incoming information from long-term memory. Why retain something that is not true? Why retain a lie?

Any information is perceived by the left brain as a potential threat to the stability and security of the individual. "Stick with the tried and the true." "Better to be safe than sorry." The blocking mechanism of the left brain is to erase information so that many trials are necessary until the left brain fatigues and concludes, "I give up! If you insist, I will store the information in long-term retention, even though it is against my better judgment."

The ape experiments

A famous example of "insight" that produces one-trial learning is the ape experiments by the Gestalt psychologists. A hungry ape is in a cage with several boxes placed at random on the floor along with a stick.Hanging from the ceiling are bananas high enough to be out of reach.

The ape tries to leap up and grab the fruit many times but is unsuccessful. He sits on a box and seems to be puzzled. Finally, he picks up the stick and tries a number of times to knock the bananas down, but that strategy does not work.

He sits on a box again and looks perplexed. Then, suddenly, he stands up and places several boxes on top of each other, climbs up and retrieves the bananas. In an instant flash of recognition, he seems to see a cause-effect connection between boxes on top of each other and access to the food.

Wrap it up: Benefits for kids in math classes

My hypothesis is this: If parents and teachers prepare youngsters with my interpretation of why multiplication works, it may be easier for the brain to assimilate "nitty gritty" procedures in arithmetic.

With my interpretation, they have a chance to see cause- effect relationships for procedures that now "do not make sense." They have a chance for "insight"—the marvelous "Aha, I get it!" response rather than settling for the instructor asserting: "Just do it because I'm telling you it works! Don't ask me why! Just memorize it!"

Part of this article is excerpted from my new book, **The Weird and Wonderful World of Mathematical Mysteries: Conversations with famous scientists and mathematicians**.

Footnotes

Footnote 1: Another interesting mystery. The numbers 1 and 2 do not behave like numbers which follow them, such as 3, 4, and 5.

1 + 1 = 2 but 1 times 1 = 1	(When we multiply, there is a <u>decrease</u> in value.)
2 + 2 = 4 and 2 times 2 = 4	(When we multiply, there is no <u>increase</u> in value.)
but,	
3 + 3 = 6 and 3 times 3 = 9	(When we multiply, there is an <u>increase</u> in value.)
4 + 4 = 8 and 4 times 4 = 16	(When we multiply, there is an <u>increase</u> in value.)
5 + 5 = 10 and 5 times 5 = 25	(When we multiply, there is an <u>increase</u> in value.)

My conclusion is that the numbers 1 and 2 behave differently compared with the numbers to follow such as 3, 4 and 5. How come?

Footnote 2: The writer invites readers to suggest an explanation that children can understand.
Any other comments or suggestions are welcome!
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Recommended follow-up reading

Asher, James J. (2005). **A Simplified Guide to Statistics for Non-mathematicians:** How to organize a successful research project. Los Gatos, CA: Sky Oaks Productions, Inc.

Asher, James J. (2005). **The Weird and Wonderful World of Mathematical Mysteries:** Conversations with famous scientists and mathematicians. Los Gatos, CA: Sky Oaks Productions, Inc.

Asher, James J. (2002). **Brainswitching: Learning on the right side of the brain**. Los Gatos, CA: Sky Oaks Productions, Inc.

Asher, James J. (2000). **The Super School: Teaching on the right side of the brain**. Los Gatos, CA: Sky Oaks Productions, Inc.

Buxton, Laurie (1984). **Mathematics For Everyone**. New York: Schocken Books

Articles you can print out from the web

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Asher, James J. **The Myth of Algebra**

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